Derivation of 1D Kalman Filter with Covariance Discussion

System Model

We assume a scalar state evolving as:

$$x_k = x_{k-1} + w_k, \quad w_k \sim \mathcal{N}(0, Q)$$

and measurements:

$$z_k = x_k + v_k, \quad v_k \sim \mathcal{N}(0, R)$$

Here:

- Q: process noise variance
- R: measurement noise variance
- x_k: true state
- z_k: measurement

Goal

Estimate the posterior distribution:

$$p(x_k | z_1, z_2, \dots, z_k)$$

given prior $p(x_{k-1} | z_1, \dots, z_{k-1})$.

Predict Step (Prior)

From the state model:

$$\hat{x}_{k|k-1} = \hat{x}_{k-1}, \qquad P_{k|k-1} = P_{k-1} + Q$$

Interpretation: $P_{k|k-1}$ is the variance of the estimation error before incorporating the new measurement. It grows by Q because process noise adds uncertainty.

Update Step (Posterior)

Measurement likelihood:

$$z_k | x_k \sim \mathcal{N}(x_k, R)$$

By Bayes' rule:

$$p(x_k | z_k) \propto p(z_k | x_k)\, p(x_k | z_{1:k-1})$$

Both are Gaussian:

$$\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \times \mathcal{N}(z_k; x_k, R)$$

The product of two Gaussians is another Gaussian with:

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k (z_k - \hat{x}_{k|k-1})$$

$$P_k = (1 - K_k) P_{k|k-1}$$

where the Kalman gain is:

$$K_k = \frac{P_{k|k-1}}{P_{k|k-1} + R}$$

Discussion of P_k

- $P_k = \mathrm{Var}(x_k - \hat{x}_k)$: the posterior error variance after the update.
- Initially, P_0 is large (high uncertainty).
- After each measurement, P_k decreases because we gain confidence.
- If Q is large, uncertainty grows faster between updates.
- If R is large, measurements are unreliable, so P_k remains larger.

Summary of 1D Kalman Filter Equations

$$
\begin{aligned}
\text{Predict:}\quad & \hat{x}_{k|k-1} = \hat{x}_{k-1}, \quad P_{k|k-1} = P_{k-1} + Q \\
\text{Update:}\quad & K_k = \frac{P_{k|k-1}}{P_{k|k-1} + R} \\
& \hat{x}_k = \hat{x}_{k|k-1} + K_k(z_k - \hat{x}_{k|k-1}) \\
& P_k = (1 - K_k)P_{k|k-1}
\end{aligned}
$$

Table 1: Advantages, Disadvantages, and Limitations of Kalman Filter

Advantages	**Disadvantages**	**Limitations**
Optimal estimation under Gaussian noise	Assumes linear dynamics	Not robust to model errors
Real-time recursive operation	Requires accurate system model	Cannot handle nonlinearities directly
Efficient sensor fusion	Sensitive to initialization	Computational complexity grows with state dimension
Predictive capability for missing data	Gaussian noise assumption	Requires full observability
Computationally efficient for small systems	Numerical stability issues for large systems	Performance degrades with incorrect noise statistics
Provides uncertainty estimates (covariance)	Divergence possible if model is wrong	Limited to linear systems without modifications